JOURNAL OF DIFFERENTIAL EQUATIONS 59, 206–228 (1985)

A Semigroup Approach to Nonlinear Nonautonomous Neutral Functional Differential Equations

JANET DYSON

Mansfield College, Oxford, OXI 3TF, England

AND

ROSANNA VILLELLA-BRESSAN*

Istituto di Analisi e Meccanica, Università di Padova, Padova, Italy

Received October 31, 1983; revised April 2, 1984

A class of nonlinear nonautonomous neutral functional differential equations is studied by associating with it an evolution equation in the space of initial data, the space $W^{1,1}$. Existence and regularity results are proved.

1. INTRODUCTION

We study the existence and regularity of solutions of a class of nonlinear nonautonomous neutral functional differential equations by associating with it an evolution equation in the space of initial data.

The equation we consider is

$$\dot{x}(t) = F(t, x_t), \qquad x_s = \varphi \in W^{1,1}(-r, 0; X), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{1.1}$$

where $x: [s-r, T] \to X$, $s \geqslant 0$, $0 < r < \infty$, is the delay and X is a Banach space. $F: [0, T] \times W^{1,1}(-r, 0; X) \to X$ and $x_t \in W^{1,1}(-r, 0; X)$ is the history of x at time t defined pointwise by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$.

Suppose that for every $\varphi \in W^{1,1}$ and $s \geqslant 0$ (1.1) has the unique solution $x(s, \varphi)(t)$. Then if we define $U(t, s)\varphi = x_t(s, \varphi)$, $U(t, s)$ is an evolution operator in $W^{1,1}$ which is a translation and thus has a generator $A(t)$, $t \geqslant 0$, defined by

$$A(t)\varphi = -\varphi', \qquad D(A(t)) = \{\varphi \in W^{2,1}: \varphi'(0) = F(t, \varphi)\}. \tag{1.2}$$

* Partially supported by grants from the Accademia dei Lincei and the Ministero della Pubblica Istruzione.

0022-0396/85 $3.00

So we are led to associate with (1.1) the evolution equation

$$\frac{d}{dt}u(t)+A(t)\,u(t)=0, \qquad u(s)=\varphi, \qquad t\geqslant s\geqslant 0. \tag{1.3}$$

Conversely, under suitable hypotheses on F, it will be shown that $A(t)$ generates an evolution operator $U(t, s)$ and that $U(t, s)\,\varphi$ gives the segments of solution of (1.1).

In his book [10], Hale describes this approach for the case in which Eq. (1.1) is linear, autonomous and the initial data φ belongs to $C(-r, 0; R^n)$. More precisely, he considers equations of the type

$$\frac{d}{dt}(x(t)-f(x_t))=g(x_t), \qquad x_0=\varphi\in C(-r, 0; R^n), \tag{1.4}$$

where f and g are linear continuous operators from $C(-r, 0; R^n)$ to R^n. In a recent paper [14] Plant uses the nonlinear semigroup theory to study the nonlinear version of (1.4) in a general Banach space. He considers the equation

$$\dot{x}(t)=G(x_t), \qquad x_0=\varphi\in C^1(-r, 0; X), \tag{1.1$'$}$$

where $G\colon C^1(-r, 0; X)\to X$ is Lipschitz continuous. For the semigroup approach to neutral equations see also Burns, Cliff and Amillo Gil [1], Kunisch [11], Salomon [15] and the bibliography there.

The results by Hale and Plant on autonomous neutral equations and results by the authors [6] and by the second author and Webb [16] on nonautonomous functional and functional differential equations suggest that we study the evolution equation (1.3) using the theory of nonlinear evolution operators developed by Crandall and Pazy [4] and Evans [7]. However, we find that if we set our problem in $C^1(-r, 0; X)$, as Hale and Plant do, or in $W^{1,p}(-r, 0; X)$, with $p>1$, the conditions required to apply the theory in [4] cannot be satisfied, however much regularity is required on the t-dependence. We will discuss this point in Section 6. So we are led to set Eq. (1.1) in the space $W^{1,1}$. An analogous situation was discussed in [16] for a nonautonomous functional equation and the space $L^1(-r, 0; X)$.

We denote by $|\cdot|$ the norm in X and by $\|\cdot\|_{1,1}$ the norm in $W^{1,1}=W^{1,1}(-r, 0; X)$, and suppose that F satisfies the following hypotheses:

(H.1) For all $t\in[0, T]$, $F\colon[0, T]\times W^{1,1}\to X$ is Lipschitz continuous, that is,

$$|F(t, \varphi)-F(t, \psi)|\leqslant\gamma(t)\,\|\varphi-\psi\|_{1,1}$$

for some $\gamma(t)\in R$ and for all φ, $\psi\in W^{1,1}$; $\gamma(t)$ is bounded.

(H.2) There exists a continuous function h: $[0, T] \to X$ which is of bounded variation and a monotone increasing function L: $[0, \infty) \to [0, \infty)$ such that

$$|F(t_1, \varphi) - F(t_2, \varphi)| \leqslant |h(t_1) - h(t_2)| \; L(\|\varphi\|_{1,1})$$

for $0 \leqslant t_1, t_2 \leqslant T$ and $\varphi \in W^{1,1}$.

We shall prove that if F satisfies conditions (H.1) and (H.2), then $A(t)$ generates an evolution operator, $U(t, s)$, in $W^{1,1}$ in the sense of Crandall and Pazy. That is,

$$U(t, s)\, \varphi = \lim_{n \to \infty} \prod_{i=1}^{n} \left(I + \frac{t-s}{n} A\left(s + i\frac{t-s}{n} \right) \right)^{-1} \varphi \tag{1.5}$$

exists for all φ in $W^{1,1}$, the function $u(t) = U(t, s)\, \varphi$ is the unique weak solution of Eq. (1.3) and this solution depends continuously on the initial data. We then prove that $U(t, s)$ is a "translation" operator, that is, if

$$\begin{aligned} x(t) &= \varphi(t-s), & s - r \leqslant t \leqslant s \\ &= U(t, s)\, \varphi(0), & s < t \leqslant T \end{aligned} \tag{1.6}$$

then $x_t = U(t, s)\, \varphi$. This will allow us to prove that $x(t)$ is the unique strong solution of the neutral equation (1.1).

One interesting feature of this approach is that it yields regularity results. These will be discussed in Section 5.

2. Preliminaries

Let Y be a Banach space with norm $\|\cdot\|_Y$. The operator $A: D(A) \subset Y \to Y$ is said to be *accretive* if for each $\lambda > 0$ and $x, y \in D(A)$

$$\|x - y\|_Y \leqslant \|x - y + \lambda Ax - \lambda Ay\|_Y. \tag{2.1}$$

A is *m-accretive* if in addition $R(I + \lambda A) = Y$.

We shall also make use of the following equivalent formulation [5]. For $x, y \in Y$ define

$$D_+ \|x\|_Y\, y = \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\|_Y - \|x\|_Y}{\lambda}. \tag{2.2}$$

The limit exists since $(\|x + \lambda y\|_Y - \|x\|_Y)/\lambda$ is decreasing as $\lambda \to 0^+$. Then A is accretive if and only if, for all $u, v \in D(A)$,

$$D_+ \|u - v\|_Y (Au - Av) \geqslant 0. \tag{2.3}$$

In [3, Theorem 1] Crandall and Liggett show that if there is a real number ω such that $A+\omega I$ is m-accretive then A generates a semigroup $T(t)$ in the sense that, for $\varphi \in \overline{D(A)}$,

$$T(t)=\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}\varphi \tag{2.4}$$

exists and defines a strongly continuous semigroup.

We shall make use of the generalized domain $D_1(A)$ considered by Crandall in [2]. We define

$$D_1(A)=\{\varphi\in\overline{D(A)};\ T(t)\,\varphi \text{ is Lipschitz continuous in } t \text{ on bounded sets}\}.$$

Crandall proves that $D(A)\subset D_1(A)$. These domains are in fact equal if Y is reflexive [13].

We now collect some results for nonautonomous equations. The operator $U(t,s)$, $0\leqslant s\leqslant t\leqslant T$, from Y to Y is said to be an *evolution operator* if

(i) $U(t,t)\,y=y$ for all $y\in Y$ and $0\leqslant t\leqslant T$,

(ii) $U(t,s)\,U(s,\tau)=U(t,\tau)$ for $0\leqslant \tau\leqslant s\leqslant t\leqslant T$.

If in addition there is an $\omega\in R$ such that, for all $x, y\in Y$,

$$\|U(t,s)\,x-U(t,s)\,y\|_Y\leqslant e^{\omega(t-s)}\,\|x-y\|_Y,\qquad 0\leqslant s\leqslant t\leqslant T \tag{2.5}$$

then $U(t,s)$ is said to be of *type* ω.

THEOREM 2.1 (Crandall and Pazy). *Let $A(t)$, $t\in[0,T]$, satisfy the following conditions*:

(C.1) *There is an $\omega\in R$ such that $A(t)+\omega I$ is m-accretive for $t\in[0,T]$.*

Let $J_\lambda(t)=(I+\lambda A(t))^{-1}$.

(C.2) *There is a continuous function, h: $[0,T]\to Y$, which is of bounded variation, and a monotone increasing function L: $[0,\infty)\to[0,\infty)$ such that*

$$\|J_\lambda(t)\,x-J_\lambda(\tau)\,x\|_Y\leqslant\lambda\,\|h(t)-h(\tau)\|_Y\,L(\|x\|_Y) \tag{2.6}$$

for $0<\lambda<\lambda_0$, $0\leqslant t$, $\tau\leqslant T$ and $x\in Y$.

Then

$$U(t,s)\,x=\lim_{n\to\infty}\prod_{i=1}^{n}J_{(t-s)/n}\left(s+i\,\frac{(t-s)}{n}\right)x \tag{2.7}$$

exists for $x \in \overline{D(A(t))}$ *and* $0 \leqslant s \leqslant t \leqslant T$ *and* $U(t, s)$ *is an evolution operator of type* ω *on* $\overline{D(A(t))}$.

Note that (C.2) implies that both $D_1(A(t)) = D_1$ and $\overline{D(A(t))} = D$ are independent of t [7].

Now let X be a Banach space with norm $|\cdot|$. We shall work in the space of initial data $W^{1,1} = W^{1,1}(-r, 0; X)$. That is,

$W^{1,1} = \{\varphi \in L^1; \varphi$ is absolutely continuous, φ' exists a.e. and $\varphi' \in L^1\}$.

We shall denote the norm in $L^1 = L^1(-r, 0; X)$ by $\|\cdot\|$ and that in $W^{1,1}$ by $\|\cdot\|_{1,1}$. So

$$\|\varphi\|_{1,1} = \|\varphi\| + \|\varphi'\|. \tag{2.8}$$

We say that the evolution operator $U(t, s)$ is a *translation* on $Z \subset W^{1,1}$ if, for all $0 \leqslant s \leqslant t \leqslant T$ and $\varphi \in Z$,

$$U(t, s)\, \varphi \in Z \qquad \text{and} \qquad U(t, s)\, \varphi = x_t, \tag{2.9}$$

where

$$\begin{aligned} x(t) &= \varphi(t-s), && s-r \leqslant t < s \\ &= U(t, s)\, \varphi(0), && s \leqslant t \leqslant T. \end{aligned} \tag{2.10}$$

LEMMA 2.1. *$U(t, s)$ is a translation on $W^{1,1}$ if and only if, for all $\varphi \in W^{1,1}$,*

$$U(t, s)\, \varphi(\theta) = \varphi(\theta + t - s) \qquad \textit{for } s - r \leqslant t + \theta < s. \tag{2.11}$$

Proof. By continuity (2.11) holds for $\theta = -(t-s)$, that is,

$$U(t, s)\, \varphi(s-t) = \varphi(0) \qquad \text{for all } \varphi \in W^{1,1},\ t \geqslant s.$$

Hence, if $t + \theta > s$

$$U(t, s)\, \varphi(\theta) = U(t, t+\theta)(U(t+\theta, s)\, \varphi)(\theta) = U(t+\theta, s)\, \varphi(0).$$

So

$$\begin{aligned} U(t, s)\, \varphi(\theta) &= \varphi(\theta + t - s) && \text{if} \quad \theta + t - s \leqslant 0 \\ &= U(t+\theta, s)\, \varphi(0) && \text{if} \quad \theta + t - s > 0. \end{aligned}$$

That is,

$$U(t, s)\, \varphi(\theta) = x(t+\theta) \qquad \text{for all } \theta \in [-r, 0],$$

where $x(t)$ is as in (2.10). So $U(t, s)\, \varphi = x_t$.

LEMMA 2.2 *If an evolution operator $U(t, s)$ in $W^{1,1}$ is a translation on a dense subset Z, then it is a translation on $W^{1,1}$.*

Proof. Let $\varphi \in W^{1,1}$, then there exist $\varphi_n \in Z$ such that φ_n tends to φ in $W^{1,1}$. Let $t \leqslant s+r$. Then $U(t, s)\,\varphi_n(\theta)$ tends to $U(t, s)\,\varphi(\theta)$ in $L^1(s-r, -t; X)$. But $U(t, s)\,\varphi_n(\theta) = \varphi_n(\theta + t - s)$ if $\theta + t - s < 0$ and $\varphi_n(\theta + t - s)$ tends to $\varphi(\theta + t - s)$ in $L^1(s-r, -t; X)$.

Hence $U(t, s)\,\varphi(\theta) = \varphi(\theta + t - s)$ for $\theta + t - s < 0$ and the result follows from the previous lemma.

Throughout K will be an absolute constant whose value may change at each appearance.

3. The Autonomous Equation

In this section we consider the autonomous neutral equation in X

$$\dot{x}(t) = G(x_t), \qquad x_0 = \varphi, \tag{1.1'}$$

where $G\colon W^{1,1} \to X$ is Lipschitz continuous, that is:

(H′) There exists a constant γ such that

$$|G(\varphi_1) - G(\varphi_2)| \leqslant \gamma\, \|\varphi_1 - \varphi_2\|_{1,1} \qquad \text{for all } \varphi_1, \varphi_2 \in W^{1,1}$$

and the initial data φ belongs to $W^{1,1}$.

We study (1.1′) by associating with it the evolution equation in $W^{1,1}$

$$\frac{du}{dt} + Au = 0, \qquad u(0) = \varphi, \tag{3.1}$$

where A is the operator

$$A\varphi = -\varphi', \qquad D(A) = \{\varphi \in W^{2,1};\ \varphi'(0) = G(\varphi)\}.$$

We prove that A generates a semigroup of operators $T(t)$, and that the (in general weak) solution of (3.1), $T(t)\,\varphi$, gives the segments of solution of (1.1′). More precisely we prove that $x(t)$ defined by

$$\begin{aligned} x(t) &= \varphi(t), && -r \leqslant t \leqslant 0 \\ &= T(t)\,\varphi(0), && t > 0 \end{aligned} \tag{3.2}$$

is a solution of (1.1′).

A crucial point of the proof is to show that $T(t)$ is a translation semigroup. This can be done directly from the exponential formula (1.5) as we will see in the next section. But it can also be done by relating $T(t)$ to the semigroup $\tilde{T}(t)$ generated in C^1 by the operator $\tilde{A}$ defined by

$$\tilde{A}\varphi = -\varphi', \qquad D(\tilde{A}) = D(A) \cap C^2.$$

This approach is interesting as it shows how our results are related to those of Plant [14] and it also gives information about the semigroup $T(t)$. Hence we give a brief description of it.

We first prove that $A + (1+\gamma)I$ is m-accretive. Consider operators of the type

$$Bu = -u', \qquad D(B) \subset W^{1,1}. \tag{3.3}$$

We characterize the sets $D(B)$ such that B is accretive in $W^{1,1}$. It is an easy application of an analogous result of Martello [12] in the space L^1.

PROPOSITION 3.1. *The operator $B + \beta I$, $\beta \in R$, is accretive if and only if*

$$|u_1(0) - u_2(0)| + |u_1'(0) - u_2'(0)| \leqslant |u_1(-r) - u_2(-r)| + |u_1'(-r) - u_2'(-r)| + \beta \, \|u_1 - u_2\|_{1,1} \tag{3.4}$$

for all $u_1, u_2 \in D(B)$.

Proof. It is proved in Proposition 1.1 of [12] that $D\,\|u\|\,(-u') = |u(-r)| - |u(0)|$ for all $u \in W^{1,1}$, and so, if $u \in W^{2,1}$ we have also $D\,\|u'\|\,(-u'') = |u'(-r)| - |u'(0)|$. Hence

$$\begin{aligned} D\,\|u\|_{1,1}\,(-u') &= D\,\|u\|\,(-u') + D\,\|u'\|\,(-u'') \\ &= |u(-r)| + |u'(-r)| - (|u(0)| + |u'(0)|) \end{aligned}$$

and so also

$$D\,\|u\|_{1,1}(-u' + \beta u) = |u(-r)| + |u'(-r)| - (|u(0)| + |u'(0)|) + \beta\,\|u\|_{1,1}. \tag{3.5}$$

The result now follows from Eq. (2.3), taking $u = u_1 - u_2$.

PROPOSITION 3.2. $A + (1+\gamma)I$ *is accretive.*

Proof. Let $u_1, u_2 \in D(A)$. Then

$$|u_1'(0) - u_2'(0)| = |G(u_1) - G(u_2)| \leqslant \gamma\,\|u_1 - u_2\|_{1,1},$$

$$\begin{aligned} |u_1(0) - u_2(0)| &= \left| u_1(-r) - u_2(-r) + \int_{-r}^{0} (u_1'(\theta) - u_2'(\theta))\,d\theta \right| \\ &\leqslant |u_1(-r) - u_2(-r)| + \|u_1' - u_2'\|. \end{aligned}$$

Hence

$$\begin{aligned}|u_1(0)-u_2(0)| &+ |u_1'(0)-u_2'(0)|\\ &\leqslant |u_1(-r)-u_2(-r)| + \|u_1'-u_2'\| + \gamma\, \|u_1-u_2\|_{1,1}\\ &\leqslant |u_1(-r)-u_2(-r)| + |u_1'(-r)-u_2'(-r)| + (1+\gamma)\, \|u_1-u_2\|_{1,1},\end{aligned}$$

and the result follows from Proposition 3.1.

PROPOSITION 3.3. *$A+(1+\gamma)I$ is m-accretive in $W^{1,1}$.*

Proof. We have to prove that $R(I+\lambda A)=W^{1,1}$ for λ small. Let $\psi \in W^{1,1}$ and φ be such that $\varphi - \lambda\varphi' = \psi$, that is,

$$\varphi(\theta) = e^{\theta/\lambda}\varphi(0) + \int_\theta^0 (e^{(\theta-s)/\lambda}/\lambda)\, \psi(s)\, ds.$$

We prove that we can choose $\varphi(0)$ so that $\varphi \in D(A)$, that is, $\varphi(0)=\psi(0)+\lambda G(\varphi)$. Set

$$H\colon X \to X, \qquad H(x) = \psi(0) + \lambda G\left(e^{\theta/\lambda}x + \int_\theta^0 (e^{(\theta-s)/\lambda}/\lambda)\, \psi(s)\, ds\right),$$

then

$$|H(x_1)-H(x_2)| \leqslant \lambda\gamma\, \|e^{\theta/\lambda}(x_1-x_2)\|_{1,1} = \lambda\gamma(\lambda+1)(1-e^{-r/\lambda})\, |x_1-x_2|,$$

and for λ small, H is a contraction. Therefore it has a unique fixed point $\bar{x}$, and if $\bar{\varphi} = e^{\theta/\lambda}\bar{x} + \int_\theta^0 (e^{(\theta-s)/\lambda}/\lambda)\, \psi(s)\, ds$, then $(I+\lambda A)\,\bar{\varphi} = \psi$.

It follows that A is the generator, in the sense of Crandall and Liggett, of a semigroup, $T(t)$, of type $1+\gamma$ in $\overline{D(A)}$. We prove now that $T(t)$ is a semigroup in $W^{1,1}$.

PROPOSITION 3.4. *$D(A)$ is dense in $W^{1,1}$. In fact*

$$\lim_{\lambda \to 0} (I+\lambda A)^{-1}\psi = \psi \tag{3.6}$$

for all $\psi \in W^{1,1}$.

Proof. Set $\varphi = (I+\lambda A)^{-1}\psi$. Let A_0 be the operator

$$A_0 u = -u', \qquad D(A_0) = \{u \in W^{2,1}, u'(0)=0\}.$$

A_0 is the generator of a linear strongly continuous semigroup in $W^{1,1}$, therefore $\lim_{\lambda\to 0}(I+\lambda A_0)^{-1}\psi = \psi$. We have

$$\|\varphi-\psi\|_{1,1}=\left\|\psi(\theta)-e^{\theta/\lambda}\varphi(0)-\int_\theta^0 (e^{(\theta-s)/\lambda}/\lambda)\,\psi(s)\,ds\right\|_{1,1}$$
$$=\left\|e^{\theta/\lambda}\psi(0)-e^{\theta/\lambda}\phi(0)+\psi(\theta)-e^{\theta/\lambda}\psi(0)-\int_\theta^0 (e^{(\theta-s)/\lambda}/\lambda)\,\psi(s)\,ds\right\|_{1,1}$$
$$\leqslant \|e^{\theta/\lambda}\|_{1,1}\,|\varphi(0)-\psi(0)|+\|\psi-(I+\lambda A_0)^{-1}\,\psi\|_{1,1},$$

also

$$|\varphi(0)-\psi(0)|=\lambda\,|G(\varphi)|\leqslant\lambda\,|G(\varphi)-G(\psi)|+\lambda\,|G(\psi)|$$
$$\leqslant\lambda\gamma\,\|\varphi-\psi\|_{1,1}+\lambda\,|G(\psi)|.$$

Hence

$$\|\varphi-\psi\|_{1,1}\leqslant\frac{1}{1-\lambda\gamma(1+\lambda)(1-e^{-r/\lambda})}\,(\lambda(\lambda+1)(1-e^{-r/\lambda})\,|G(\psi)|)$$
$$+\|\psi-(I+\lambda A_0)^{-1}\,\psi\|_{1,1},$$

and the right-hand side tends to zero as $\lambda\to 0$.

We consider now the operator $\tilde{A}$ in C^1 defined by

$$\tilde{A}\varphi=-\varphi',\qquad D(\tilde{A})=D(A)\cap C^2=\{\varphi\in C^2;\,\varphi'(0)=G(\varphi)\}$$

and prove that $\tilde{A}$ satisfies the conditions required in [14]. More precisely we prove that

PROPOSITION 3.5. *Let G satisfy condition* (H$'$). *Then there exist $\sigma\geqslant 0$ and $\gamma_\sigma<1$ such that*

$$|G(\varphi_1)-G(\varphi_2)|\leqslant\gamma r\,\|\varphi_1-\varphi_2\|_\infty+\gamma_\sigma\sup_{-r\leqslant\theta\leqslant 0} e^{-\sigma\theta}\,|\varphi_1'(\theta)-\varphi_2'(\theta)| \tag{3.7}$$

for all φ_1, $\varphi_2\in C^1$, where $\|\varphi\|_\infty=\sup_{\theta\in[-r,0]}|\varphi(\theta)|$.

Proof. Let $p\in R$ be such that $0<p<r$ and $\gamma p<1$. Then

$$|G(\varphi_1)-G(\varphi_2)|\leqslant\gamma r\,\|\varphi_1-\varphi_2\|_\infty+\gamma\int_{-r}^0|\varphi_1'(\theta)-\varphi_2'(\theta)|\,d\theta$$
$$\leqslant\gamma r\,\|\varphi_1-\varphi_2\|_\infty+\gamma(r-p)\,e^{-\sigma p}$$
$$\times\sup_{-r\leqslant\theta\leqslant -p}\{e^{-\sigma\theta}\,|\varphi_1'(\theta)-\varphi_2'(\theta)|\}$$
$$+\gamma p\sup_{-p\leqslant\theta\leqslant 0}\{e^{-\sigma\theta}\,|\varphi_1'(\theta)-\varphi_2'(\theta)|\}$$
$$\leqslant\gamma r\,\|\varphi_1-\varphi_2\|_\infty+(\gamma(r-p)\,e^{-\sigma p}+\gamma p)$$
$$\times\sup_{-r\leqslant\theta\leqslant 0}\{e^{-\sigma\theta}\,|\varphi_1'(\theta)-\varphi_2'(\theta)|\}.$$

It is now easy to verify that (3.7) is satisfied for any

$$\sigma > \max\left\{0, \frac{1}{p}\log\frac{\gamma(r-p)}{1-\gamma p}\right\}$$

with $\gamma_\sigma = \gamma(r-p)\, e^{-\sigma p} + \gamma p$.

It is proved in [14] that if G satisfies (3.7) with $\gamma_\sigma < 1$, then $\tilde{A}$ generates a semigroup of translations $\tilde{T}(t)$, in the set $E = \{\varphi \in C^1, \varphi'(0) = G(\varphi)\}$; and that if we choose any $\varphi \in E$ and set

$$\begin{aligned} \tilde{x}(t) &= \varphi(t), \qquad && -r \leqslant t \leqslant 0 \\ &= \tilde{T}(t)\,\varphi(0), && t > 0 \end{aligned}$$

then $\tilde{x}(t)$ is a solution of (1.1′); such a solution is continuously differentiable.

Note that, as $E \supset D(A)$, E is dense in $W^{1,1}$. Moreover, for all $\varphi \in E$ we have $(I + \lambda A)^{-1}\varphi = (I + \lambda\tilde{A})^{-1}\varphi$, and so

$$\begin{aligned} \tilde{T}(t)\,\varphi &= C^1 - \lim_{n\to\infty}\left(I + \frac{t}{n}\tilde{A}\right)^{-n}\varphi \\ &= W^{1,1} - \lim_{n\to\infty}\left(I + \frac{t}{n}\tilde{A}\right)^{-n}\varphi = W^{1,1} - \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n}\varphi \\ &= T(t)\,\varphi. \end{aligned}$$

Hence E is flow invariant for $T(t)$, that is, $T(t)\,E \subset E$.

From Lemma 2.2 it follows that

PROPOSITION 3.6. *$T(t)$ is a semigroup of translations in $W^{1,1}$.*

PROPOSITION 3.7. *For $\varphi \in W^{1,1}$ set*

$$\begin{aligned} x(t) &= \varphi(t), \qquad && t \in [-r, 0] \\ &= T(t)\,\varphi(0), && t > 0. \end{aligned}$$

Then $x(t)$ satisfies the neutral equation

$$\dot{x}(t) = G(x_t) \qquad a.e.\ t \geqslant 0,\ x_0 = \varphi.$$

Proof. Let $\varphi \in W^{1,1}$, then there exist $\varphi_n \in E$ such that $\varphi_n \to^{W^{1,1}} \varphi$. From above $x(t)$ is a solution of (1.1′) for $\varphi \in E$ and also $T(t)$ is a translation.

Hence from the definition of translation

$$\frac{d}{d\theta} T(t)\,\varphi_n(\theta) = \frac{d}{dt} T(t+\theta)\,\varphi_n(0) = G(T(t+\theta)\,\varphi_n).$$

However, $(d/d\theta)\,T(t)\,\varphi_n(\theta) \to^{L^1} (d/d\theta)\,T(t)\,\varphi(\theta)$.
Also

$$\int_{-r}^{0} |G(T(t+\theta)\,\varphi_n) - G(T(t+\theta)\,\varphi)|\,d\theta$$

$$\leqslant \int_{-r}^{0} \gamma\,\|T(t+\theta)\,\varphi_n - T(t+\theta)\,\varphi\|_{1,1}\,d\theta$$

$$\leqslant \gamma \int_{-r}^{0} e^{(1+\gamma)(t+\theta)}\,\|\varphi_n - \varphi\|_{1,1}\,d\theta$$

so that $G(T(t+\theta)\,\varphi_n) \to^{L^1} G(T(t+\theta)\,\varphi)$.
Thus $(d/d\theta)\,T(t)\,\varphi(\theta) = G(T(t+\theta)\,\varphi)$ a.e. so that $(d/dt)\,T(t)\,\varphi(0) = G(T(t)\,\varphi)$ a.e. $t \geqslant 0$.

We now prove that $x(t)$ is the unique solution of (1.1′). We say that $x(t)$ is a *strong solution* of (1.1′) if

(i) $x(t)$ is absolutely continuous on $[0, T]$,

(ii) $x(t)$ is differentiable a.e. on $(0, T)$ and satisfies (1.1′) a.e.

Then

PROPOSITION 3.8. *Let G satisfy* (H′). *Then there is at most one strong solution of* (1.1′).

Proof. Let $x(t)$, $y(t)$ be strong solutions of (1.1′). Then, for any $r_1 > 0$, if $t \in [0, r_1]$

$$\|x'_t - y'_t\| = \int_{-r}^{0} |x'(t+\theta) - y'(t+\theta)|\,d\theta$$

$$= \int_{0}^{t} |x'(\tau) - y'(\tau)|\,d\tau$$

$$= \int_{0}^{t} |G(x_\tau) - G(y_\tau)|\,d\tau$$

$$\leqslant \gamma \int_{0}^{t} \|x_\tau - y_\tau\|_{1,1}\,d\tau.$$

Also

$$\|x_t - y_t\| = \int_{-r}^{0} |x(t+\theta) - y(t+\theta)|\, d\theta$$

$$\leqslant \gamma \int_0^t \int_0^\tau |G(x_\sigma) - G(y_\sigma)|\, d\sigma\, d\tau$$

$$\leqslant \gamma r \int_0^t \|x_\sigma - y_\sigma\|_{1,1}\, d\sigma.$$

Thus $\|x_t - y_t\|_{1,1} \leqslant \gamma(1+r_1)\int_0^t \|x_\sigma - y_\sigma\|_{1,1}\, d\sigma$, so that, for $t \in [0, r_1]$, $\|x_t - y_t\|_{1,1} = 0$ by Gronwall's inequality and thus $x(t) = y(t)$.

These results are finally summarised in

THEOREM 3.1. *Let G satisfy* (H′). *Then A generates a semigroup $T(t)$ of type $1+\gamma$ in $W^{1,1}$. Set*

$$x(t) = \varphi(t), \qquad -r \leqslant t \leqslant 0.$$
$$= T(t)\,\varphi(0), \qquad t > 0,$$

then $x(t)$ is the unique strong solution of the neutral equation

$$\dot{x}(t) = G(x_t), \qquad x_0 = \varphi$$

for all $\varphi \in W^{1,1}$. If $\varphi \in E = \{\varphi \in C^1;\ \varphi'(0) = G(\varphi)\}$, then the solution is continuously differentiable.

4. THE NONAUTONOMOUS EQUATION

We now turn to the nonautonomous neutral equation in X

$$\dot{x}(t) = F(t, x_t), \qquad 0 \leqslant s \leqslant t \leqslant T, \qquad x_s = \varphi, \tag{1.1}$$

where F: $[0, T] \times W^{1,1} \to X$ satisfies conditions (H.1) and (H.2). Define the operators $A(t)$, $0 \leqslant t \leqslant T$, by

$$A(t)\,\varphi = -\varphi', \qquad D(A(t)) = \{\varphi \in W^{2,1};\ \varphi'(0) = F(t, \varphi)\}. \tag{4.1}$$

We prove that $A(t)$ generates an evolution operator $U(t, s)$ in the sense of Crandall and Pazy and that $U(t, s)\,\varphi$ gives the segments of solution of (1.1). In the previous section the equivalent result was proved first in the flow-invariant subset $E \subset W^{1,1}$ and the result extended to $W^{1,1}$ using a density argument. The subset E now depends on t, so cannot be used, but we find that $D_1(A)$ plays a similar role.

We first verify that the operators $A(t)$ satisfy the conditions of the theorem of Crandall and Pazy. Let $\gamma = \sup_{\tau \in [0,T]} \gamma(\tau)$, then by Propositions 3.2 and 3.3 $A(t) + (\gamma + 1) I$ is m-accretive for $t \in [0, T]$, so (C.1) is satisfied. That (C.2) is also satisfied is a consequence of the following proposition.

PROPOSITION 4.1. *Suppose that F satisfies* (H.1) *and* (H.2). *Then there is a monotonically increasing function L_1: $[0, \infty) \to [0, \infty)$ such that, for $0 < \lambda < \lambda_0$, $0 \leqslant t, \tau \leqslant T$ and $\varphi \in W^{1,1}$,*

$$\|J_\lambda(t)\, \varphi - J_\lambda(\tau)\, \varphi\|_{1,1} \leqslant \lambda\, |h(t) - h(\tau)|\, L_1(\|\varphi\|_{1,1}),$$

where h is the function defined in (H.2).

Proof. We have

$$J_\lambda(t)\, \varphi(\theta) - \lambda (J_\lambda(t)\, \varphi)'\, (\theta) = \varphi(\theta) \tag{4.2}$$

so that

$$J_\lambda(t)\, \varphi(\theta) = e^{\theta/\lambda} J_\lambda(t)\, \varphi(0) + \int_\theta^0 (e^{(\theta - s)/\lambda}/\lambda)\, \varphi(s)\, ds, \tag{4.3}$$

where $J_\lambda(t)\, \varphi(0) \in D(A(t))$, so that, using (4.2),

$$J_\lambda(t)\, \varphi(0) = \varphi(0) + \lambda F(t, J_\lambda(t)\, \varphi), \tag{4.4}$$

Thus

$$\begin{aligned}
&\|J_\lambda(t)\, \varphi - J_\lambda(\tau)\, \varphi\|_{1,1} \\
&\quad = \|e^{\theta/\lambda} (J_\lambda(t)\, \varphi(0) - J_\lambda(\tau)\, \varphi(0))\|_{1,1} \\
&\quad = (1 + \lambda)(1 - e^{-r/\lambda})\, |J_\lambda(t)\, \varphi(0) - J_\lambda(\tau)\, \varphi(0)| \\
&\quad = (1 + \lambda)(1 - e^{-r/\lambda})\, \lambda\, |F(t, J_\lambda(t)\, \varphi) - F(\tau, J_\lambda(\tau)\, \varphi)| \\
&\quad \leqslant K\lambda \{ |F(t, J_\lambda(t)\, \varphi) - F(t, J_\lambda(\tau)\, \varphi)| + |F(t, J_\lambda(\tau)\, \varphi) - F(\tau, J_\lambda(\tau)\, \varphi)| \} \\
&\quad \leqslant K\lambda \{ \gamma\, \|J_\lambda(t)\, \varphi - J_\lambda(\tau)\, \varphi\|_{1,1} + |h(t) - h(\tau)|\, L(\|J_\lambda(\tau)\, \varphi\|_{1,1}) \}
\end{aligned}$$

so that, for λ small enough,

$$\|J_\lambda(t)\, \varphi - J_\lambda(\tau)\, \varphi\|_{1,1} \leqslant K\lambda\, |h(t) - h(\tau)|\, L(\|J_\lambda(\tau)\, \varphi\|_{1,1}).$$

Finally, to estimate $\|J_\lambda(\tau)\, \varphi\|_{1,1}$ set $\tau = 0$ above, so

$$\|J_\lambda(t)\, \varphi\|_{1,1} \leqslant \|J_\lambda(0)\, \varphi\|_{1,1} + K\lambda\, |h(t) - h(0)|\, L(\|J_\lambda(0)\, \varphi\|_{1,1}).$$

However, for some fixed $\varphi_0 \in D(A(0))$

$$\|J_\lambda(0)\,\varphi\|_{1,1} \leqslant \|J_\lambda(0)\,\varphi - J_\lambda(0)\,\varphi_0\|_{1,1} + \|J_\lambda(0)\,\varphi_0 - \varphi_0\|_{1,1} + \|\varphi_0\|_{1,1}$$
$$\leqslant \frac{1}{1-\lambda(1+\gamma)}\,\|\varphi - \varphi_0\|_{1,1}$$
$$+ \frac{\lambda}{1-\lambda(1+\gamma)}\,\|A(0)\,\varphi_0\|_{1,1} + \|\varphi_0\|_{1,1}$$

and the result is proved by taking $t = \tau$.

Thus the conditions of the theorem of Crandall and Pazy are satisfied and $A(t)$ generates an evolution operator $U(t, s)$ in the sense that, if $\lambda = (t-s)/n$,

$$\lim_{n \to \infty} \prod_{i=1}^{n} J_\lambda(s + i\lambda)\,\varphi = U(t, s)\,\varphi$$

for all $\varphi \in \overline{D(A(t))}$. By Proposition 3.4 $\overline{D(A(t))} = W^{1,1}$ for all t.

We now have to prove that $U(t, s)\,\varphi$ is a translation and that $x(t) = U(t, s)\,\varphi(0)$ is the solution of (1.1) for all $\varphi \in W^{1,1}$. This is proved first for $\varphi \in D_1 = D_1(A(t))$ directly from the definition of $U(t, s)$. The result will then be extended to $\varphi \in W^{1,1}$ using a density argument as in Section 3.

Following Flaschka and Leitman we define an operator $A_1: C \to C$ $(C = C(-r, 0; X))$ by

$$A_1\varphi = -\varphi', \qquad D(A_1) = \{\varphi \in C, \varphi(0) = 0, \varphi' \in C\} \tag{4.5}$$

and $J_{1,\lambda}$ by

$$J_{1,\lambda}\varphi(\theta) = (I + \lambda A_1)^{-1}\varphi(\theta) \tag{4.6}$$
$$= \int_\theta^0 (e^{(\theta - s)/\lambda}/\lambda)\,\varphi(s)\,ds \tag{4.7}$$

as in (4.3).

Then although A_1 is not the generator of a semigroup Flaschka and Leitman prove that for $\varphi \in C$

$$\lim_{n \to \infty} J^n_{1,t/n}\varphi(\theta) = \varphi(\theta + t), \qquad -r \leqslant \theta < -t \tag{4.8}$$
$$= 0, \qquad -t < \theta \leqslant 0.$$

From now on we will set $\lambda = (t-s)/n$.

PROPOSITION 4.2. *For all* $\varphi \in W^{1,1}$

$$\prod_{i=1}^{n} J_\lambda(s+i\lambda)\,\varphi(\theta) = \sum_{j=0}^{n-1} \frac{(n\sigma)^j}{j!} e^{-n\sigma} \left(\varphi(0) + \sum_{m=1}^{n-j} \lambda F\left(s+m\lambda, \prod_{i=1}^{m} J_\lambda(s+i\lambda)\,\varphi\right)\right) + J_{1,\lambda}^n \varphi(\theta), \tag{4.9}$$

where $\sigma = -\theta/(t-s)$.

Proof. From (4.3)

$$J_\lambda(t)\,\varphi(\theta) = e^{\theta/\lambda} J_\lambda(t)\,\varphi(0) + J_{1,\lambda}\varphi(\theta)$$

so by induction

$$\begin{aligned}\prod_{i=1}^{n} J_\lambda(s+i\lambda)\,\varphi(\theta) &= \sum_{j=0}^{n-1} J_{1,\lambda}^j(\exp \tau/\lambda)(\theta) \prod_{i=1}^{n-j} J_\lambda(s+i\lambda)\,\varphi(0) + J_{1,\lambda}^n\varphi(\theta) \\ &= \sum_{j=0}^{n-1} J_{1,\lambda}^j(\exp \tau/\lambda)(\theta) \left\{ \sum_{m=1}^{n-j} \left(\prod_{i=1}^{m} J_\lambda(s+i\lambda)\,\varphi(0) \right.\right. \\ &\qquad \left.\left. - \prod_{i=1}^{m-1} J_\lambda(s+i\lambda)\,\varphi(0) \right) + \varphi(0) \right\} + J_{1,\lambda}^n \varphi(\theta).\end{aligned}$$

However,

$$J_{1,\lambda}^j(\exp \tau/\lambda)(\theta) = \frac{n^j}{j!} \left(-\frac{\theta}{t-s}\right)^j e^{\theta/\lambda}$$

and from (4.4)

$$\prod_{i=1}^{m} J_\lambda(s+i\lambda)\,\varphi(0) - \prod_{i=1}^{m-1} J_\lambda(s+i\lambda)\,\varphi(0) = \lambda F\left(s+m\lambda, \prod_{i=1}^{m} J_\lambda(s+i\lambda)\,\varphi\right).$$

Equation (4.9) follows immediately.

We now require the following lemmas:

LEMMA 4.1. *For* $\varphi \in W^{1,1}$, $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned}\lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{(n\sigma)^j}{j!} e^{-n\sigma} \left(\varphi(0) + \int_s^{t-j((t-s)/n)} F(\tau, U(\tau,s)\,\varphi)\,d\tau\right) &= 0 \qquad \text{if} \quad \sigma > 1 \\ &= \varphi(0) + \int_0^{t+\theta} F(\tau, U(\tau,s)\,\varphi)\,d\tau \qquad \text{if} \quad 0 \leqslant \sigma < 1.\end{aligned} \tag{4.10}$$

Proof. If $u\colon [0, \infty) \to X$ is bounded and continuous, then the Poisson probability distribution satisfies

$$\lim_{n \to \infty} e^{-n\sigma} \sum_{j=0}^{\infty} u\left(\frac{j}{n}\right) \frac{(n\sigma)^j}{j!} = u(\sigma) \tag{4.11}$$

[8, p. 220]. If we define

$$\begin{aligned} u(\sigma) &= \varphi(0) + \int_s^{t-\sigma(t-s)} F(\tau, U(\tau, s)\varphi)\, d\tau && \text{if} \quad 0 \leqslant \sigma < 1 \\ &= 0 && \text{if} \quad \sigma \geqslant 1 \end{aligned}$$

then (4.11) also holds for this u (which has a single jump discontinuity) so that (4.10) is true.

LEMMA 4.2. *If* $\sigma \geqslant 0$, $\varphi \in W^{1,1}$, $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned} &\lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{(n\sigma)^j}{j!} e^{-n\sigma} \\ &\quad \times \left\{ \sum_{m=1}^{n-j} \lambda F(s + m\lambda, U(s + m\lambda, s)\varphi) - \int_s^{t - j((t-s)/n)} F(\tau, U(\tau, s)\varphi)\, d\tau \right\} = 0. \end{aligned} \tag{4.12}$$

Proof. Set $g(\tau) = F(\tau, U(\tau, s)\varphi)$. This is a continuous function of τ so given $\varepsilon > 0$ there is an N such that for $n \geqslant N$ and each $0 \leqslant j \leqslant n-1$

$$\left| \sum_{m=1}^{n-j} \frac{(t-s)}{n} g(s + m\lambda) - \int_s^{t - j((t-s)/n)} g(\tau)\, d\tau \right| < \varepsilon.$$

If in (4.11) we now set

$$\begin{aligned} u(\sigma) &= 1 && \text{if} \quad 0 \leqslant \sigma < 1 \\ &= 0 && \text{if} \quad \sigma > 1 \end{aligned}$$

we see that

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} e^{-n\sigma} \frac{(n\sigma)^j}{j!} \begin{aligned} &= 1 && \text{if} \quad 0 \leqslant \sigma < 1 \\ &= 0 && \text{if} \quad \sigma \geqslant 1. \end{aligned} \tag{4.13}$$

The result follows.

LEMMA 4.3. *If $\sigma \geqslant 0$, then for $\varphi \in D_1$, $0 \leqslant s \leqslant t \leqslant T$*

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{(n\sigma)^j}{j!} e^{-n\sigma} \sum_{m=1}^{n-j} \lambda \times \left[F(s+m\lambda, U(s+m\lambda, s)\varphi) - F\left(s+m\lambda, \prod_{i=1}^{m} J_\lambda(s+i\lambda)\varphi\right)\right] = 0. \tag{4.14}$$

Proof. Define

$$\rho(r) = \sup\{|h(t)-h(\tau)|; 0 \leqslant t, \tau \leqslant T, |t-\tau| \leqslant r\}$$

so that ρ is nondecreasing and $\lim_{r \to 0} \rho(r) = \rho(0) = 0$. Then by [4, Proposition 2.5] there is a constant K such that, for $\varphi \in D_1$ and $1 \leqslant m \leqslant n$,

$$\begin{aligned} \left\| U(s+m\lambda, s)\varphi - \prod_{i=1}^{m} J_\lambda(s+i\lambda)\varphi \right\|_{1,1} &\leqslant Km \frac{(t-s)}{n} \left\{ m^{-1/2} + \rho\left(m^{3/4} \left(\frac{t-s}{n}\right)\right)\right\} \\ &\leqslant K \left\{ \frac{(t-s)}{n} m^{1/2} + m \frac{(t-s)}{n} \rho\left(\frac{t-s}{n^{1/4}}\right)\right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\left| \sum_{m=1}^{n-j} \lambda F(s+m\lambda, U(s+m\lambda, s)\varphi) - F\left(s+m\lambda, \prod_{i=1}^{m} J_\lambda(s+i\lambda)\varphi\right)\right| \\ &\leqslant \gamma \sum_{m=1}^{n} \lambda \left\| U(s+m\lambda, s)\varphi - \prod_{i=1}^{m} J_\lambda(s+i\lambda)\varphi \right\|_{1,1} \\ &\leqslant K\gamma \sum_{m=1}^{n} \left\{ \left(\frac{t-s}{n}\right)^2 m^{1/2} + m \left(\frac{t-s}{n}\right)^2 \rho\left(\frac{t-s}{n^{1/4}}\right)\right\} \\ &\leqslant K\gamma \left\{ \left(\frac{t-s}{n}\right)^2 n^{3/4} + n^2 \left(\frac{t-s}{n}\right)^2 \rho\left(\frac{t-s}{n^{1/4}}\right)\right\}. \end{aligned}$$

This tends to zero as $n \to \infty$ and the result follows as in Lemma 4.2.

THEOREM 4.1. *For $t \in [s, T]$ and $\varphi \in D_1$*

$$\begin{aligned} U(t,s)\varphi(\theta) &= \varphi(\theta+t-s) && \text{if} \quad -r \leqslant \theta \leqslant -(t-s) \\ &= \varphi(0) + \int_s^{t+\theta} F(\tau, U(\tau, s)\varphi)\, d\tau && \text{if} \quad -(t-s) < \theta \leqslant 0. \end{aligned} \tag{4.15}$$

Proof. From Proposition 4.2

$$
\begin{aligned}
\prod_{i=1}^{n} & J_\lambda(s+i\lambda)\,\varphi(\theta) \\
&= \sum_{j=0}^{n-1} \frac{(n\sigma)^j}{j!} e^{-n\sigma} \left\{ \left(\varphi(0) + \int_s^{t-j((t-s)/n)} F(\tau, U(\tau, s)\,\varphi)\,d\tau \right) \right. \\
&\quad - \left(\int_s^{t-j((t-s)/n)} F(\tau, U(\tau, s)\,\varphi)\,d\tau - \sum_{m=1}^{n-j} \lambda F(s+m\lambda, U(s+m\lambda, s)\,\varphi) \right) \\
&\quad - \left(\sum_{m=1}^{n-j} \lambda \left(F(s+m\lambda, U(s+m\lambda, s)\,\varphi) \right.\right. \\
&\quad \left.\left.\left. - F\left(s+m\lambda, \prod_{i=1}^{m} J_\lambda(s+i\lambda)\,\varphi \right) \right) \right) \right\} \\
&\quad + J_{1,\lambda}^{n}(\theta),
\end{aligned}
$$

where $\sigma = -\theta/(t-s)$. Lemmas 4.1, 4.2 and 4.3 together with (4.8) show that $\prod_{i=1}^{n} J_\lambda(s+i\lambda)\,\varphi(\theta)$ converges pointwise to the continuous function on the right-hand side of (4.15). However, this product also converges in $W^{1,1}$ to $U(t,s)\,\varphi$ and thus (4.15) is true.

Theorem 4.1 shows that, for $\varphi \in D_1$, $U(t,s)\,\varphi$ is a translation. It then follows from Lemma 2.2 that $U(t,s)\,\varphi$ is a translation for $\varphi \in W^{1,1}$.

THEOREM 4.2. *Let F satisfy* (H.1) *and* (H.2). *For $\varphi \in W^{1,1}$ set*

$$
\begin{aligned}
x(t) &= \varphi(t-s), && s-r \leqslant t \leqslant s \\
&= U(t,s)\,\varphi(0), && s \leqslant t \leqslant T.
\end{aligned}
\tag{4.16}
$$

Then $x(t)$ is the unique solution of

$$\dot{x}(t) = F(t, x_t) \qquad \text{a.e. } s \leqslant t \leqslant T,\ x_s = \varphi,$$

and if $\varphi \in D_1$ this solution is continuously differentiable.

Proof. Differentiating (4.15), we see that, for $\varphi \in D_1$,

$$\frac{d}{dt} U(t,s)\,\varphi(0) = F(t, U(t,s)\,\varphi). \tag{4.17}$$

Since D_1 is dense in $W^{1,1}$ and $U(t,s)\,\varphi$ is a translation we may use the same argument as Proposition 3.7 to show that, for $\varphi \in W^{1,1}$, (4.17) holds almost everywhere. The uniqueness follows as in Proposition 3.8.

5. Regularity

In Section 3 we found that

$$T(t): \overline{D(\tilde{A})}^{C^1} \to \overline{D(\tilde{A})}^{C^1}.$$

From Theorem 4.2 we have a similar result in the nonautonomous case. For, if $\varphi \in D_1$ and $t-s \geqslant r$

$$U(t, s)\, \varphi \in \{\psi \in C^1; \psi'(0) = F(t, \psi)\}.$$

That is, $U(t, s)\colon D_1 \to \overline{D(A(t))}^{C^1}$, so that, in particular, for initial data in D_1 solutions are continuously differentiable if $t > s$ and continuously differentiable at $t = s$ if the initial function is in $\overline{D(A(s))}^{C^1}$.

We now give a characterization of D_1.

Theorem 5.1. *Let $F(\tau, \cdot)$ satisfy* (H.1). *Then the function $\varphi \in W^{1,1}$ is in $D_1(A(\tau))$ if and only if there is a constant K such that the following three conditions hold for all $t \in [0, r]$*:

$$\int_{-r}^{-t} |\varphi(\theta + t) - \varphi(\theta)|\, d\theta \leqslant Kt, \tag{5.1}$$

$$\int_{-r}^{-t} \left| \frac{d}{d\theta} \varphi(\theta + t) - \frac{d}{d\theta} \varphi(\theta) \right| d\theta \leqslant Kt, \tag{5.2}$$

$$\int_{-t}^{0} \left| \frac{d}{d\theta} \varphi(\theta) \right| d\theta \leqslant Kt. \tag{5.3}$$

Proof. For convenience we suppress τ. Let $S(t)$ be the semigroup generated by A. We recall that $\varphi \in D_1(A)$ if and only if for $T > 0$, $t, s \in [0, T]$ there is a constant K such that

$$\|S(t)\, \varphi - S(s)\, \varphi\|_{1,1} \leqslant K\, |t - s|. \tag{5.4}$$

It is sufficient to consider this only for $|t - s| \leqslant r$, for, if $|t - s| > r$,

$$\|S(t)\, \varphi - S(s)\, \varphi\|_{1,1} \leqslant \|S(t)\, \varphi\|_{1,1} + \|S(s)\, \varphi\|_{1,1} \leqslant K \leqslant K \frac{|t - s|}{r}$$

since $S(t)\, \varphi$ is continuous in t.

Now, for $t \in [0, r]$

$$\begin{aligned} &\|S(t)\,\varphi-\varphi\|_{1,1} \\ &\quad=\int_{-r}^{-t} |\varphi(\theta+t)-\varphi(\theta)|\, d\theta+\int_{-r}^{-t} \left|\frac{d}{d\theta}\,\varphi(\theta+t)-\frac{d}{d\theta}\,\varphi(\theta)\right| d\theta \\ &\quad\quad+\int_{-t}^{0} |S(t)\,\varphi(\theta)-\varphi(\theta)|\, d\theta+\int_{-t}^{0} \left|\frac{d}{d\theta}\,S(t)\,\varphi(\theta)-\frac{d}{d\theta}\,\varphi(\theta)\right| d\theta, \end{aligned} \tag{5.5}$$

However, the continuity of $\varphi(\theta)$ and $S(t)\,\varphi$ in θ and t, respectively, together with the translation property imply that

$$\int_{-t}^{0} |S(t)\,\varphi(\theta)-\varphi(\theta)|\, d\theta \leqslant Kt \tag{5.6}$$

and

$$\int_{-t}^{0} \left|\frac{d}{d\theta}\,S(t)\,\varphi(\theta)\right| d\theta=\int_{-t}^{0} |F(S(t+\theta)\,\varphi)|\, d\theta \leqslant Kt. \tag{5.7}$$

Thus, if $\varphi \in D_1(A)$, (5.1), (5.2) and (5.3) hold.

Suppose now that (5.1), (5.2) and (5.3) hold. Then, if $|t-s| \leqslant r$, where $t, s \in [0, T]$

$$\|S(t)\,\varphi-S(s)\,\varphi\|_{1,1} \leqslant e^{T(\gamma+1)}\,\|S(t-s)\,\varphi-\varphi\|_{1,1}$$

so that using (5.5), (5.6) and (5.7), $\varphi \in D_1(A)$.

6. Choice of the Space of Initial Data

In order to satisfy the hypotheses of the Crandall and Pazy theorem, the operators $A(t)$ must satisfy the condition (C.2). This condition implies that the sets $D_1(A(t))$ must be independent of t, and, since $D(A(t)) \subset D_1(A(t)) \subset \overline{D(A(t))}$, that $\overline{D(A(t))}$ is also independent of t. Working in $W^{1,1}$ we saw in the previous section that indeed $D_1(A(t))$ is independent of F and hence is independent of t.

In [14] Eq. (1.1′) was studied in the space of initial data C^1 and it was shown that

$$\overline{D(\tilde{A})}=\{\varphi \in C^1,\ \varphi(0)=G(\varphi)\}.$$

It follows that it is not possible to use the Crandall and Pazy theory to study the nonautonomous equation (1.1) in C^1 as the solution would be generated by the operators

$$\tilde{A}(t)\,\varphi = -\varphi', \qquad D(\tilde{A}(t)) = \{\varphi \in C^2,\ \varphi'(0) = F(t, \varphi)\}$$

so that $\overline{D(\tilde{A}(t))} = \{\varphi \in C^1,\ \varphi'(0) = F(t, \varphi)\}$ and thus depends on t.

There is a similar problem in the space $W^{1,p}$, $p > 1$.

Suppose that X is a Hilbert space and associate with (1.1) the family of operators in $W^{1,p}(-r, 0; X)$, $p > 1$,

$$A_p(t)\,\varphi = -\varphi', \qquad D(A_p(t)) = \{\varphi \in W^{2,p},\ \varphi'(0) = F(t, \varphi)\}.$$

As $W^{1,p}(-r, 0; X)$ is reflexive $D_1(A_p(t))$ coincides with $D(A_p(t))$ and thus depends on t; so again $A_p(t)$ cannot satisfy (C.2).

Thus $W^{1,1}$ is the natural space for studying the nonautonomous equation (1.1).

7. An Example

We now apply our results to the integro-differential equation

$$\begin{aligned} \dot{x}(t) &= f(t) + \int_{t-r}^{t} K_1(t, \tau, x(\tau))\,d\tau + \int_{t-r}^{t} K_2(t, \tau, \dot{x}(\tau))\,d\tau, \\ &\qquad 0 \leqslant s \leqslant t \leqslant T \qquad\qquad (7.1) \\ x(t) &= \varphi(t-s), \qquad s - r \leqslant t \leqslant s, \end{aligned}$$

where $\varphi \in W^{1,1}$ and $f\colon [0, T] \to X$, $K_i\colon [0, T] \times [-r, T] \times X \to X$ satisfy the following hypotheses.

f is Lipschitz continuous in $[0, T]$ with Lipschitz constant L. (7.2)

There are constants C_1, C_2 such that for all $t_1, t_2 \in [0, T]$, $\tau \in [-r, T]$, $x \in X$ $|K_i(t_1, \tau, x) - K_i(t_2, \tau, x)| \leqslant C_i\,|t_1 - t_2|\,|x|$. (7.3)

There are constants D_1, D_2 such that for all $t \in [0, T]$, $\tau_1, \tau_2 \in [-r, T]$, $x \in X$ $|K_i(t, \tau_1, x) - K_i(t, \tau_2, x)| \leqslant D_i\,|\tau_1 - \tau_2|\,|x|$. (7.4)

There are bounded functions $\gamma_1, \gamma_2\colon [0, T] \to R$ such that for all $t \in [0, T]$ $\tau \in [-r, T]$, $x_1, x_2 \in X$ $|K_i(t, \tau, x_1) - K_i(t, \tau, x_2)| \leqslant \gamma_i(t)\,|x_1 - x_2|$. (7.5)

Define $F\colon [0, T] \times W^{1,1} \to X$ by

$$F(t, \varphi) = f(t) + \int_{t-r}^{t} K_1(t, \tau, \varphi(\tau - t))\, d\tau + \int_{t-r}^{t} K_2(t, \tau, \varphi'(\tau - t))\, d\tau \tag{7.6}$$

for all $t \in [0, T]$, $\varphi \in W^{1,1}$.

We verify that F satisfies (H.1) and (H.2). To prove (H.1) we have, for all $t \in [0, T]$, from (7.5) that

$$\begin{aligned} |F(t, \varphi) - F(t, \psi)| &\leqslant \int_{t-r}^{t} \gamma_1(t)\, |\varphi(\tau - t) - \psi(\tau - t)|\, d\tau \\ &\quad + \int_{t-r}^{t} \gamma_2(t)\, |\varphi'(\tau - t) - \psi'(\tau - t)|\, d\tau \\ &\leqslant \max\{\gamma_1(t), \gamma_2(t)\}\, \|\varphi - \psi\|_{1,1}. \end{aligned}$$

For (H.2) we have, from (7.2), (7.3), (7.4), that, for all $t_1, t_2 \in [0, T]$, $\varphi \in W^{1,1}$,

$$\begin{aligned} &|F(t_1, \varphi) - F(t_2, \varphi)| \\ &\quad \leqslant |f(t_1) - f(t_2)| \\ &\qquad + \left| \int_{t_1-r}^{t_1} K_1(t_1, \tau, \varphi(\tau - t_1))\, d\tau - \int_{t_1-r}^{t_1} K_1(t_2, \tau - t_1 + t_2, \varphi(\tau - t_1))\, d\tau \right| \\ &\qquad + \left| \int_{t_1-r}^{t_1} K_2(t_1, \tau, \varphi'(\tau - t_1))\, d\tau - \int_{t_1-r}^{t_1} K_2(t_2, \tau - t_1 + t_2, \varphi'(\tau - t_1))\, d\tau \right| \\ &\quad \leqslant L\, |t_1 - t_2| + C_1\, |t_1 - t_2| \int_{t_1-r}^{t_1} |\varphi(\tau - t_1)|\, d\tau \\ &\qquad + D_1\, |t_1 - t_2| \int_{t_1-r}^{t_1} |\varphi(\tau - t_1)|\, d\tau + C_2\, |t_1 - t_2| \int_{t_1-r}^{t_1} |\varphi'(\tau - t_1)|\, d\tau \\ &\qquad + D_2\, |t_1 - t_2| \int_{t_1-r}^{t_1} |\varphi'(\tau - t_1)|\, d\tau \\ &\quad \leqslant K\, |t_1 - t_2|\, (1 + \|\varphi\|_{1,1}). \end{aligned}$$

Thus Theorem 4.2 applies to Eq. (7.1) with F as in (7.6).

References

1. J. E. Burns, E. M. Cliff, and J. M. Amillo Gil, Nonlinear neutral functional differential equations in product spaces, *in* "Ordinary and Partial Differential Equations" (W. N. Everitt and B. D. Sleeman, Eds.), pp. 118–134, Springer-Verlag, New York/Berlin, 1982.

2. M. G. Crandall, A generalized domain for semigroup generators, *Proc. Amer. Math. Soc.* **37** (1973), 434–440.
3. M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, *Amer. J. Math.* **93** (1971), 265–298.
4. M. G. Crandall and A. Pazy, Nonlinear evolution equations in Banach spaces, *Israel J. Math.* **11** (1972), 57–94.
5. G. Da Prato, Applications croissantes et équations d'évolution dans les espaces de Banach, *in* "Istitutiones Mathematicae," Vol. 2, Academic Press, New York, 1976.
6. J. Dyson and R. Villella-Bressan, Functional differential equations and nonlinear evolution operators, *Proc. Roy. Soc. Edinburgh Sect. A* **75** (1976), 223–234.
7. L. C. Evans, Nonlinear evolution equations in an arbitrary Banach space, *Israel J. Math.* **26** (1977), 1–42.
8. W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. II, Wiley, New York, 1966.
9. H. Flaschka and M. J. Leitman, On semigroups of nonlinear operators and the solution of the functional differential equation $x'(t) = F(x_t)$, *J. Math. Anal. Appl.* **49** (1975), 649–658.
10. J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, Berlin, 1977.
11. K. Kunisch, Neutral functional differential equations in L^p spaces and averaging approximations, *J. Nonlinear Anal. TMA* **3** (1979), 419–448.
12. F. Martello, Semigruppi di traslazioni nello spazio $L^1(-r, 0; X)$, *Rend. Sem. Mat. Univ. Padova*, in press.
13. I. Miyadera, Some remarks on semigroups of nonlinear operators, *Tohoku Math. J.* **23** (1971), 443–440.
14. A. T. Plant, "Nonlinear Semigroups Generated by Neutral Functional Differential Equations," Fluid Mechanics Research Institute, Report 72, University of Essex, 1976.
15. D. Salomon, "Neutral Functional Differential Equations and Semigroups of Operators," Dynamische Systeme, Report 80, Universität Bremen, 1982.
16. R. Villella-Bressan and G. F. Webb, Nonautonomous functional equations and nonlinear evolution operators, *Rend. Sem. Mat. Univ. Padova* **71** (1984), 177–184.